Markov Chains and Search Applications

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Introduction

- Markov chains are one of the most useful stochastic processes
  - Simple and flexible
  - Pervasive
  - Low computation costs

- Many applications in economics and elsewhere
  - Google PageRank
  - Business cycle models (boom-bust cycles)
  - **Job Search Theory**

<u>Goal</u>: Become familiar with foundations of Markov processes and work through a simple application

Introduction

- Say $X \in \{1, 2, \ldots, N\}$ is a set of states. Then the process governing $X(t)$ is **Markovian** iff

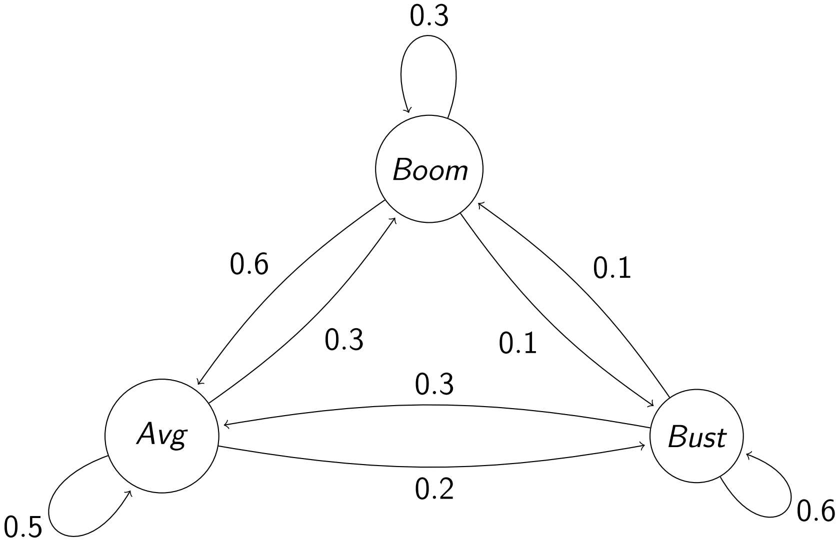
$$P(X(t_{n+1}) = j | X(t_n) = i, X(t_{n-1}) = i_{n-1} \ldots)$$
$$= P(X(t_{n+1}) = i | X(t_n) = i)$$

- A Markov process is **time homogeneous** iff

$$P(X(t) = j | X(s) = i) = P(X(t-s) = j | X(0) = i)$$

- Using time homogeneity, you can show several key properties
  - Holding times, T_i are distributed geometrically
  - Potential states are constant over time

Introduction



Discrete Time

Discrete Time Processes

- Completely characterized by transition (Markov) matrix, P
- **Transition matrix** is an $n \times n$ matrix such that
  - All elements are non-negative
  - All rows sum to 1

$$P = \begin{array}{c} \begin{array}{ccc} \mathit{Boom} & \mathit{Avg} & \mathit{Bust} \end{array} \\ \left(\begin{array}{ccc} 0.3 & 0.6 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{array} \right) \end{array} \begin{array}{l} \\ \mathit{Boom} \\ \mathit{Avg} \\ \mathit{Bust} \end{array}$$

- Gives probability of going from $i \to j$

$$\pi'_{t+1} = \pi'_t P$$

Discrete Time Processes

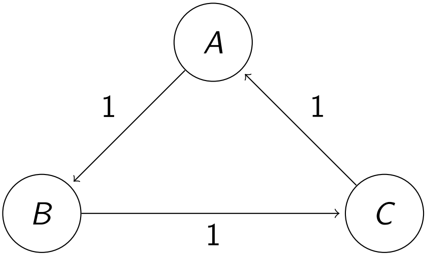
- Two central concepts to Markov chain theory are *periodicity* and *irreducibility*
  - **Periodic** if the chain cycles in a predictable way

$$LCD\ of\ D(x) \equiv \{j \geq |P^j(x, x) > 0\}$$

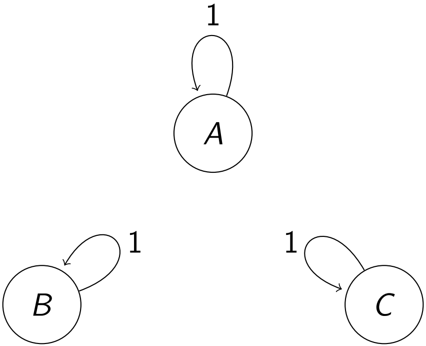
  - **Irreducible** if all states communicate, i.e.

$$\exists\ k, j\ s.t.\ P^k(x, y) > 0\ and\ P^j(y, x) > 0\ \forall\ x, y$$

Discrete Time Process

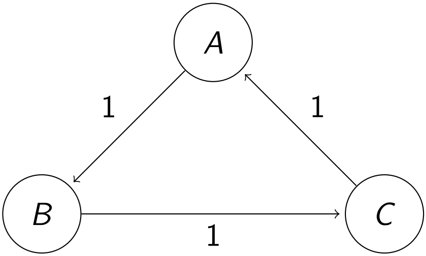


Periodic

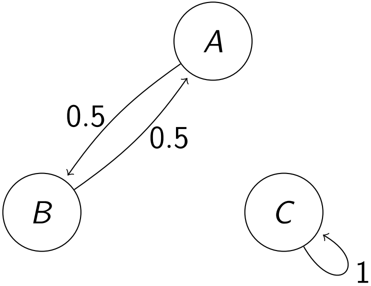


Aperiodic

Discrete Time Processses



Irreducible



Reducible

Discrete Time Processes

- Typically interested in asymptotic (stationary) distribution of processes

$$\pi' = \pi' P$$

- **Stationary distribution** gives long-run proportion of time in each state
- Getting to the asymptotic distribution where $X \in \{0, 1\}$:

$$\begin{aligned} Pr(X_{t+2} = j | X_t = i) &= P(X_{t+2} = j | X_{t+1} = 0) P(X_{t+1} = 0 | X_t = i) \\ &+ P(X_{t+2} = j | X_{t+1} = 1) P(X_{t+1} = 1 | X_t = i) \\ &= p_{0j} p_{i0}^{(1)} + p_{1j} p_{i1}^{(1)} \end{aligned}$$

Discrete Time Processes

- Thus,

$$p_{00}^{(2)} = p_{00}p_{00} + p_{10}p_{01} \qquad p_{01}^{(2)} = p_{01}p_{00} + p_{11}p_{01}$$

$$p_{10}^{(2)} = p_{00}p_{10} + p_{11}p_{11} \qquad p_{11}^{(2)} = p_{01}p_{10} + p_{11}p_{11}$$

- Notice,

$$PP = \begin{bmatrix} p_{00}p_{00} + p_{10}p_{01} & p_{01}p_{00} + p_{11}p_{01} \\ p_{00}p_{10} + p_{11}p_{11} & p_{01}p_{10} + p_{11}p_{11} \end{bmatrix}$$

Discrete Time Processes

- Iterating forward, we get the asymptotic distribution:

$$\left(\lim_{n\to\infty} P^n\right)_{ij} = \pi_j$$

- Alternatively, find the eigenvector of P' corresponding to an eigenvalue of 1

$$\lambda v = Av \;\Leftrightarrow\; \pi = P'\pi$$

- **Lemma**: If P is both irreducible and aperiodic,
  1) P has a unique stationary distribution
  2) For any initial distribution π_0, $\lim_{n\to\infty} \|\pi_0 P^n - \pi^*\| \to 0$

Example

Recall the transition matrix of a model economy:

$$P = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} \text{\textit{Boom}} & \text{\textit{Avg}} & \text{\textit{Bust}} \end{array}$$

$$P = \begin{pmatrix} 0.3 & 0.6 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \begin{array}{l} \textit{Boom} \\ \textit{Avg} \\ \textit{Bust} \end{array}$$

What is the long-run probability of the economy being in each state of growth?

$\boldsymbol{n}$	5	10	20
$\boldsymbol{P^n}$	$\begin{bmatrix} 0.2447 \\ 0.4696 \\ 0.2858 \end{bmatrix}$	$\begin{bmatrix} 0.2414 \\ 0.4656 \\ 0.2930 \end{bmatrix}$	$\begin{bmatrix} 0.2414 \\ 0.4655 \\ 0.2931 \end{bmatrix}$

Homework

Say that the transition matrix, P, is

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

Use your favorite numerical tool to find the steady state distribution using the following three methods and compare the coputation time of each:

1) Iterate on $\pi'_{t+1} = \pi'_t P$
2) The eigenvalue-eigenvector method
3) Iterate the transition matrix forward, $\left(\lim_{n\to\infty} P^n\right)_{ij} = \pi_j$

Repeat for

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.5 & 0.1 & 0.2 & 0.2 \\ 0.7 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.6 & 0.1 \end{bmatrix}$$

Continuous Time

Continuous Time Processes

- Must turn to continuous time for many phenomena
- Must rely on a differential equation rather than transition matrix
- Using time homogeneity, you can show several key properties (as before)
  1) Holding time in each state, T_i, is exponentially distributed
  2) Potential states are constant over time
  3) Probability of reaching a state in t units is constant

Continuous Time Processes

- Contiuous time processes characterized by transition *rates* not probabilities

- Let q_{ij} be the transition rate from state $i \rightarrow j$ and $v_i = \sum_{k \neq i} q_{ik}$. Then by prop. 1

$$Pr(T_i = t) = v_i e^{-v_i t} \quad Pr(X_n = j | X_{n-1} = j, \; event) = \frac{q_{ij}}{v_i}$$

- Using a Taylor series expansion,

$$Pr(T_i > h) \approx 1 - v_i h + o(h) \quad Pr(T_i \leq h) \approx v_i h + o(h)$$

Continuous Time Processes

- We can now characterize the probability of transitioning from $i \to j$

$$
\begin{aligned}
p_{ij}(t+h) &= P(X(t+h) = j | X(0) = i) \\
&= \sum_{k \in S} P(X(t+h) = j | X(h) = k) P(X(h) = k | X(0) = i \\
&= \sum_{k \in S} P(X(t) = j | X(0) = k) P(X(h) = k | X(0) = i) \\
&= \sum_{k \in S} p_{kj}(t) P(X(h) = k | X(0) = i)
\end{aligned}
$$

- Seperate out the $k = i$ term,

$$
\begin{aligned}
p_{ij}(t+h) = p_{ij}(t) P(X(h) = i | X(0) = i) + \\
\sum_{k \neq i} p_{kj}(t) P(X(h) = k | X(0) = i)
\end{aligned}
$$

Continuous Time Processes

- Now, we can use our approximate holding time probabilities from before

$$p_{ij}(t+h) = p_{ij}(t)\underbrace{(1 - v_i h + o(h))}_{\text{No transition}} + \sum_{k \neq i} p_{kj}(t)\ \underbrace{\tilde{p}_{ik}(v_i h + o(h))}_{\text{Transition from } i \to k}$$

- Rearranging, substituting $q_{ik} = v_i \tilde{p}_{ik}$, and taking $h \to 0$

$$p'_{ij}(t) = \sum_{k \neq i} q_{ik} p_{kj}(t) - v_i p_{ij}(t)$$

- Hence, we obtain a Kolmogorov differential equation

$$P'(t) = QP(t) \quad \textit{where} \quad Q = \begin{bmatrix} -v_1 & q_{12} & \dots & q_{1s} \\ q_{21} & -v_2 & \dots & q_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ q_{s1} & q_{s2} & \dots & -v_s \end{bmatrix}$$

Continuous Time Processes

- Given $P(0) = I$, we have completely characterized the process
$$P(t) = e^{tQ} \equiv \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!}$$

- Stationary distribution of CTMC given by $\pi' = \pi' P(t) \; \forall \; t$
  - Can prove this is equivalent to $\pi' Q = 0$
  - Known as *global balance equation*

- Above relationship becomes unmanageable fast

- Rely on the *embedded jump chain* given by holding times
  - Markov chain given an event occurs
  - Transforms problem to discrete chain

Continuous Time Processes

- Construct a transition matrix, $\tilde{P}$, given an event occurs
  - Implies $\tilde{P}_{ij} = \tilde{p}_{ij}$ and $\tilde{p}_{ii} = 0$
  - Stationary distribution given by $\psi' = \psi' \tilde{P}$

- Assume global balance equations satisfied; interpret ψ_j as long run proportion of transitions into state j

$$\psi_j = C\pi_j v_j \qquad\qquad \pi_j = \frac{1}{C} \cdot \frac{\psi_j}{v_j}$$

- Sum of states must be equal to 1

$$\Rightarrow \psi_j = \frac{\pi_j v_j}{\sum\limits_{i \in S} \pi_i v_i} \qquad\qquad \Rightarrow \pi_j = \frac{\psi_j / v_j}{\sum\limits_{i \in S} \psi_i / v_i}$$

Application

Application to Burdett-Mortensen

- Search assumes Poisson arrival rates
  - Markovian
  - Time homogeneous

- Unemp. and emp. job arrival rate of α_0 and α_1; wage offer distribution $F(x)$

- Jobs destroyed at exogenous rate δ

- Discount at rate r

- Reservation wage given by

$$w_R - b = [\alpha_0 - \alpha_1] \int_{w_R}^{\infty} \frac{1 - F(x)}{r + \delta + \alpha_1(1 - F(x))} dF(x)$$

Application to Burdett-Mortensen

- Algorithm:
  1) Construct generator matrix Q
  2) Construct $\tilde{P}$ from Q
  3) Guess ψ_0; I typically set $\psi_0(1) = 1$
  4) Iterate $\psi_i \tilde{P}$ until convergence
  5) Back-out π^* using above relationships

Application to Burdett-Mortensen

- Note that the you need to use a discrete approximation to $Pr(w_i)$:

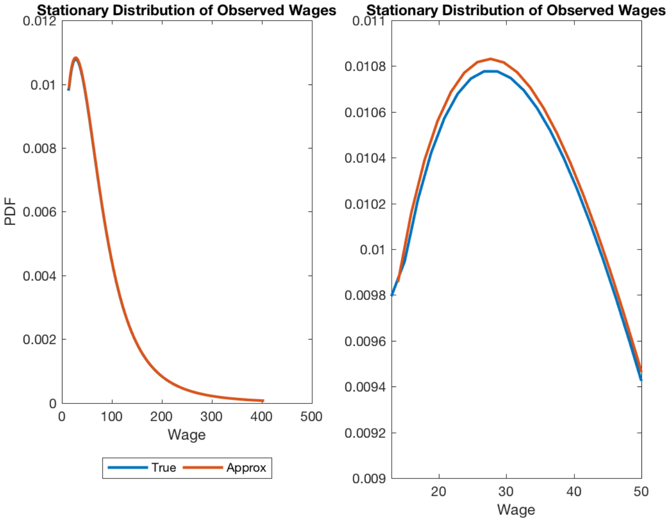
$$Pr(w = w_i) \approx F(w_i + \varepsilon) - F(w_i - \varepsilon) \quad where\ \varepsilon = 0.5\Delta w$$

- Generator matrix given by

$$Q = \begin{array}{c} \\ \left(\begin{array}{ccccc} u & w_1 & w_2 & \dots & \bar{w} \\ -\Sigma_u & \alpha_0 Pr(w_1) & \alpha_0 Pr(w_2) & \dots & \alpha_0 Pr(\bar{w}) \\ \delta & -\Sigma_1 & \alpha_1 Pr(w_2) & \dots & \alpha_1 Pr(\bar{w}) \\ \delta & 0 & -\Sigma_2 & \dots & \alpha_1 Pr(\bar{w}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta & 0 & 0 & 0 & -\Sigma_{\bar{w}} \end{array}\right) \end{array} \begin{array}{c} \\ u \\ w_1 \\ w_2 \\ \vdots \\ \bar{w} \end{array}$$

Application to Burdett-Mortensen

- I picked some numbers and used an exogenous log-normal wage distribution



Homework

Use your favorite numerical tool to approxiamte the observed wage distribution of a model of OTJ search with:

1) Arrival rates $(\alpha_0, \alpha_1) = (5, 2)$
2) Job destruction rate $\delta = 0.5$
3) Normalize $b = 0$ and $r = 0.03$
4) *Exogenous* lognormal wage distribution with $(\mu, \sigma) = (5, 0.05)$